(solution)

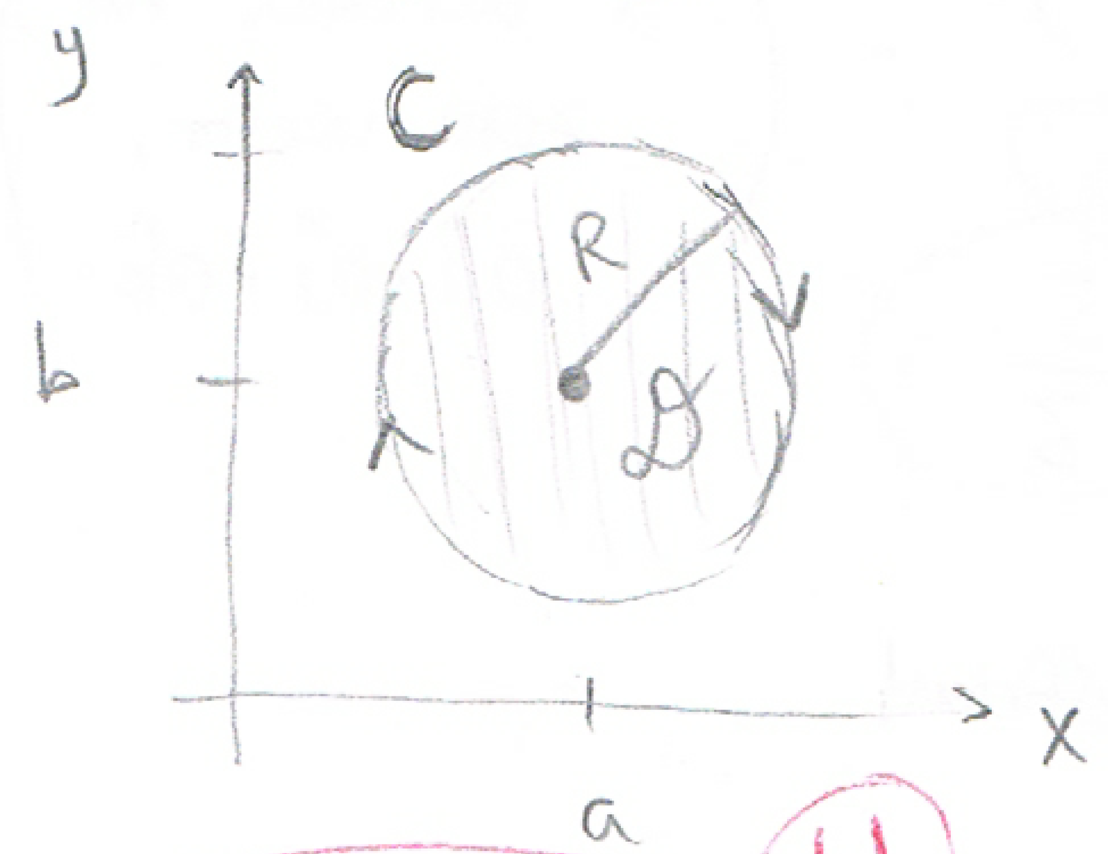
Math 2E Quiz 8 Morning - May 19th

Please write your name and ID on the front.

Show all of your work, and simplify all your answers. *There is a question on the back side.

1. Let $\mathbf{F}(x,y) = <\underbrace{y\sin x}_{P=},\ \underbrace{x-\cos x}_{Q=}>$. Let the curve C be the circle $(x-a)^2+(y-b)^2=R^2$ going clockwise. (Here, a, b, R are any real numbers). Compute $\oint_C \mathbf{F}\cdot \mathbf{dr}$.

Conceptually,



Here, $P = y\sin x$, $Q = x-\cos x$, and

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy .$$

★ P and Q are smooth so the partials are continuous.

↳ can use Green's Thm, but careful w/ signs!

★ Negatively oriented! +1

• Since C has negative orientation, $-C$ has positive orientation.

+2

$$\hookrightarrow \oint_C \vec{F}\cdot d\vec{r} = -\oint_{-C} P\,dx + Q\,dy \overset{\text{Green's Thm}}{=} -\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad +2$$

$$= -\iint_{\mathcal{D}} \left[(1+\cancel{\sin x}) - (\cancel{\sin x})\right] dA \quad +3$$

$$= -\iint_{\mathcal{D}} 1\,dA = -\text{Area}(\mathcal{D})$$

← Disk of radius R

$$= \boxed{-\pi R^2} \quad +2$$

• Incorrect Sign: −2

Here, $P = \sin y$, $Q = x\cos y + \cos z$, $R = -y\sin z$.

2. Recall: On the homework, we showed that if $\mathbf{F} = < P, Q, R >$ is conservative and P, Q, R have continuous partial derivatives, then

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}.$$

a) Verify that $\mathbf{F}(x, y, z) = < \sin y,\ x\cos y + \cos z,\ -y\sin z >$ satisfies the above equalities. (Equivalently, you can show that $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = (0, 0, 0)$.)

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$\frac{\partial Q}{\partial x} = \cos y + 0$ ✓ equal. +1

$\frac{\partial P}{\partial y} = \cos y$

$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

$\frac{\partial R}{\partial x} = 0$ ✓ equal +1

$\frac{\partial P}{\partial z} = 0$

$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$

$\frac{\partial R}{\partial y} = -\sin z$ +1 ✓ equal

$\frac{\partial Q}{\partial z} = 0 - \sin z$

(with Curl: $\nabla \times \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix} = \underbrace{(R_y - Q_z)}_{(-\sin z + \sin z)}\hat{i} + \underbrace{(P_z - R_x)}_{(0-0)}\hat{j} + \underbrace{(Q_x - P_y)}_{(\cos y - \cos y)}\hat{k} = \vec{0}$)

zero vector $0\hat{i} + 0\hat{j} + 0\hat{k}$

b) Since the components of $\mathbf{F}$ in part (a) are smooth functions, this means $\mathbf{F}$ is conservative. Let the curve C be given by $C(t) = (\sin t,\ t,\ 2t)$ from $0 \le t \le \pi/2$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = < \sin y,\ x\cos y + \cos z,\ -y\sin z >$ from part (a).

Since $\vec{F}$ ~ conservative, use the Fundamental Thm. of Line Integrals.

1) Get f s.t. $\nabla f = \vec{F}$: (more than 1 way)

$f(x,y,z) = \int -y\sin z\, dz + g(x,y)$

$= y\cos z + g(x,y)$ +1 +1

• $f_x = 0 + g_x(x,y) \overset{\text{set}}{=} \sin y$

so $g_x(x,y) = \sin y$, $g(x,y) = x\sin y + h(y)$ +1

overall: $\boxed{f(x,y,z) = y\cos z + x\sin y + K}$

• $f_y = \cos z + x\cos y + h_y(y) \overset{\text{set}}{=} x\cos y + \cos z$ so $h_y(y) = 0$, $h(y) = K$ constant +1

2) Apply Fund. Thm:

Final Point @ $t = \pi/2$: $(\sin\frac{\pi}{2}, \frac{\pi}{2}, \frac{2\pi}{2}) = (1, \frac{\pi}{2}, \pi)$

Initial Pt @ $t = 0$: $(\sin 0, 0, 2\cdot 0) = (0,0,0)$ +1

$\hookrightarrow \int_C \vec{F} \cdot d\vec{r} = f(1, \frac{\pi}{2}, \pi) - f(0,0,0) = (-\frac{\pi}{2} + 1 + K) - (K) = \boxed{1 - \frac{\pi}{2}}$ +1 +1